# Self-duality and long-wavelength behavior of the Landau-level guiding-center structure function, and the shear modulus of fractional quantum Hall fluids

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A remarkable self-duality of the "guiding-center" structure function $s(\boldsymbol{q})$ of particles in a partially-filled 2D Landau level is used to express the long-wavelength collective excitation energy of fractional quantum Hall fluids in terms of a shear modulus. A bound on the small-$\boldsymbol{q}$ behavior of $s(\boldsymbol{q})$ is given.

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Landau quantization of the kinetic energy of charged particles moving on a two-dimensional surface through which a uniform magnetic flux density $B$ passes leaves their non-commuting "guiding-center" coordinates as the residual degrees of freedom. The (static) "guiding-center structure factor" $\bar{s}(\boldsymbol{q})$ introduced by Girvin, MacDonald, and Platzman (GMP)[1] characterizes the two-particle correlation function and plays a major role in their "single-mode approximation" (SMA) treatment of collective excitations of the Laughlin incompressible fractional quantum Hall (FQH) state[2].

The problem of interacting guiding centers is essentially a *quantum geometry* problem. I will show that the guiding-center structure factor has a remarkable self-duality that (for single-component systems with no other internal degrees of freedom, such as spin-polarized electrons or spinless bosons) makes it proportional (after a rotation) to its own Fourier transform.

Using an analogy to Feynman's treatment[3] of the collective mode in superfluid $^4$He, GMP found an upper bound to the Laughlin-state collective-mode dispersion in the form $\bar{f}(\boldsymbol{q})/\bar{s}(\boldsymbol{q})$, where $\lim_{\lambda\to 0}\bar{f}(\lambda\boldsymbol{q}) \propto \lambda^4$. Using the self-duality to simplify the GMP expression for $\bar{f}(\boldsymbol{q})$, I will relate its long-wavelength limit to a "shear modulus" of the incompressible FQH fluid, which describes the energy cost of distorting the shape of the "elementary droplet" of the FQH fluid. This shape can be viewed as the shape of the "attached flux" of the "composite boson" of the fluid, and plays a central role in a "geometric field theory" description of the FQH[4].

In an idealized model, charge-$e$ particles move on a flat translationally-invariant 2D surface. The "magnetic area" through which a London flux quantum $\Phi_0 = h/e$ passes is $2\pi\ell_B^2$. A particle guiding-center $\boldsymbol{R} = R^a\boldsymbol{e}_a$, where $\{\boldsymbol{e}_a, a = 1,2\}$ are orthonormal tangent vectors of the surface, has non-commuting components

$$[R^a, R^b] = -i\epsilon^{ab}\ell_B^2, \tag{1}$$

where $\epsilon^{ab}$ is the antisymmetric (2D Levi-Civita) symbol.

In order to make any (non-topological) metric-dependence explicit, I use a (spatially) covariant notation that distinguishes displacements (with upper indices) from (reciprocal) vectors (with lower indices), with Einstein summation convention over repeated upper/lower index pairs. The tangent vectors are chosen orthonormal ($\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \eta_{ab}$) with respect to the 2D Euclidean metric $\eta_{ab}$, induced by the 3D Euclidean metric of the space in which the surface is embedded. This metric is important at atomic scales, but plays no fundamental role in FQH systems where the quantum-geometric "magnetic area" $2\pi\ell_B^2$ is much larger than the area of any atomic-scale unit cell on the surface, and provides the "ultraviolet regularization" of the continuum description.

If the energy splitting of the degenerate Landau levels dominates the interaction energies, and a single Landau level is partially occupied, the leading term in the residual effective Hamiltonian that lifts the degeneracy of the partially-filled Landau level by two-body interactions is

$$H = \int \frac{d^2\boldsymbol{q}\ell_B^2}{4\pi}\tilde{v}(\boldsymbol{q})\rho(\boldsymbol{q})\rho(-\boldsymbol{q}) \tag{2}$$

$$\rho(\boldsymbol{q}) = \sum_{\alpha\alpha'}\langle\alpha|e^{i\boldsymbol{q}\cdot\boldsymbol{R}}|\alpha'\rangle c_\alpha^\dagger c_{\alpha'}, \tag{3}$$

where $c_\alpha^\dagger$ creates a particle in state $|\alpha\rangle$, and $\alpha$ labels orbitals in an arbitrary orthonormal basis of states of the Landau level. Here $\rho(\boldsymbol{q})$ obeys the Lie algebra

$$[\rho(\boldsymbol{q}), \rho(\boldsymbol{q}')] = 2i\sin(\tfrac{1}{2}\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2)\rho(\boldsymbol{q}+\boldsymbol{q}'), \tag{4}$$

where $\boldsymbol{q}\times\boldsymbol{q}' \equiv \epsilon^{ab}q_a q_b'$, $q_a = \boldsymbol{q}\cdot\boldsymbol{e}_a$. The fundamental algebra (4) was first noticed in this context by GMP[1]. The operators $\rho(\boldsymbol{q})$ with $\boldsymbol{q} \neq 0$ are generators of area-preserving diffeomorphisms (APD) of the Landau level. A filled Landau-level is invariant under a APD, but FQH states are not[5].

The nature of the ground state of (2) depends on the (real) interaction $\tilde{v}(\boldsymbol{q}) = \tilde{v}(-\boldsymbol{q})$, which is a product of the Fourier transform of the Coulomb interaction potential between the particles, a form factor of the quantum well which binds them to the 2D surface, and a Landau orbit form-factor. The form factors ensure that $v(\boldsymbol{r})$, where

$$H = \sum_{i<j} v(\boldsymbol{R}_i - \boldsymbol{R}_j), \quad v(\boldsymbol{r}) = \int \frac{d^2\boldsymbol{q}\ell_B^2}{2\pi}\tilde{v}(\boldsymbol{q})e^{i\boldsymbol{q}\cdot\boldsymbol{r}}, \tag{5}$$

is finite. For general $\tilde{v}(\boldsymbol{q})$, the Hamiltonian (2), has both translational and 2D inversion symmetry ($\boldsymbol{R} \to -\boldsymbol{R}$).

The interaction also has rotational invariance if

$$\tilde{v}(\boldsymbol{q}) = \tilde{v}(q_g), \quad q_g^2 \equiv g^{ab} q_a q_b, \tag{6}$$

where $g^{ab}$ is the inverse of a positive-definite unimodular (determinant = 1) metric $g_{ab}$; this will only occur if the shape of the Landau orbits are congruent with the shape of the Coulomb equipotentials around a point charge on the surface. In practice, this only happens when there is an atomic-scale three-fold or four-fold rotation axis normal to the surface, and no "tilting" of the magnetic field relative to this axis, in which case $g_{ab} = \eta_{ab}$.

I will assume that translational symmetry is unbroken, so $\langle c^\dagger_\alpha c_{\alpha'}\rangle = \nu\delta_{\alpha\alpha'}$, where $\nu$ is the "filling factor" of the Landau level. (In a 2D system, this will always be true at finite temperatures, but may break down as $T \to 0$). Then $\langle \rho(\boldsymbol{q})\rangle = 2\pi\nu\delta^2(\boldsymbol{q}\ell_B)$. Note that the fluctuation $\delta\rho(\boldsymbol{q}) = \rho(\boldsymbol{q}) - \langle\rho(\boldsymbol{q})\rangle$ also obeys the algebra (4). I will define a guiding-center structure factor $s(\boldsymbol{q}) = s(-\boldsymbol{q})$ by

$$\tfrac{1}{2}\langle\{\delta\rho(\boldsymbol{q}), \delta\rho(\boldsymbol{q}')\}\rangle = 2\pi s(\boldsymbol{q})\delta^2(\boldsymbol{q}\ell_B + \boldsymbol{q}'\ell_B). \tag{7}$$

This is a structure factor defined per flux quantum, and is given in terms of the GMP structure factor $\bar{s}(\boldsymbol{q})$ of Ref.[1] (defined per particle) by $s(\boldsymbol{q}) = \nu\bar{s}(\boldsymbol{q})$. I also define $s^a(\boldsymbol{q}) \equiv \partial s(\boldsymbol{q})/\partial q_a$, $s^{ab}(\boldsymbol{q}) \equiv \partial^2 s(\boldsymbol{q})/\partial q_a\partial q_b$, *etc.*

In the "high-temperature limit" where $|v(\boldsymbol{r})| \ll k_BT$ for all $\boldsymbol{r}$, but $k_BT$ remains much smaller than the gap between Landau levels, the guiding centers become completely uncorrelated, with $\langle c^\dagger_\alpha c_{\alpha'} c^\dagger_\beta c_{\beta'}\rangle - \langle c^\dagger_\alpha c_{\alpha'}\rangle\langle c^\dagger_\beta c_{\beta'}\rangle \to s_\infty \delta_{\alpha\beta'}\delta_{\beta\alpha'}$, with $s_\infty = \nu + \xi\nu^2$, where $\xi = -1$ if the particles are spin-polarized fermions, and $\xi = +1$ if they are bosons (which may be relevant for cold-atom systems). Note that for all temperatures, $\lim_{\lambda\to\infty} s(\lambda\boldsymbol{q}) = s_\infty$, while $s(0) = \lim_{\lambda\to 0} s(\lambda\boldsymbol{q}) = k_BT/\left(\partial^2 f(T,\nu)/\partial\nu^2\big|_T\right)$, where $f(T,\nu)$ is the free energy per flux quantum. $s(0)$ vanishes at $T = 0$, and at all $T$ if $\tilde{v}(\lambda\boldsymbol{q})$ diverges as $\lambda \to 0$; the high temperature expansion at fixed $\nu$, for $\tilde{\boldsymbol{r}}_{\boldsymbol{q}} \equiv \boldsymbol{e}_a\epsilon^{ab}q_b\ell_B^2$, is

$$\frac{s(\boldsymbol{q}) - s_\infty}{(s_\infty)^2} = -\left(\frac{\tilde{v}(\boldsymbol{q}) + \xi v(\tilde{\boldsymbol{r}}_{\boldsymbol{q}})}{k_BT}\right) + O\left(\frac{1}{T^2}\right). \tag{8}$$

The correlation energy per flux quantum is given by

$$\bar{\varepsilon} = \int \frac{d^2\boldsymbol{q}\ell_B^2}{4\pi}\tilde{v}(\boldsymbol{q})\left(s(\boldsymbol{q}) - s_\infty\right). \tag{9}$$

The fundamental duality of the structure function (already apparent in (8), and derived below) is

$$s(\boldsymbol{q}) - s_\infty = \xi\int \frac{d^2\boldsymbol{q}'\ell_B^2}{2\pi} e^{i\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2}\left(s(\boldsymbol{q}') - s_\infty\right). \tag{10}$$

This is valid for a structure function calculated using *any* translationally-invariant density-matrix, and assumes that no additional degrees of freedom (*e.g.*, spin, valley, or layer indices) distinguish the particles.

Consider the equilibrium state of a system with temperature $T$ and filling factor $\nu$ with the Hamiltonian (2). The free energy of this state is formally given by $F[\hat{\rho}_{\text{eq}}]$, where $\hat{\rho}_{\text{eq}}(T,\nu)$ is the equilibrium density-matrix $Z^{-1}\exp(-H/k_BT)$ and $F[\hat{\rho}]$ is the functional

$$F[\hat{\rho}] = \mathrm{Tr}\left(\hat{\rho}(H + k_BT\log\hat{\rho})\right), \tag{11}$$

which, for fixed $\nu$, is minimized when $\hat{\rho} = \hat{\rho}_{\text{eq}}$. The APD corresponding to a shear is $R^a \to R^a + \epsilon^{ab}\gamma_{bc}R^c$, parametrized by a symmetric tensor $\gamma_{ab} = \gamma_{ba}$. Let $\hat{\rho}(\gamma) = U(\gamma)\hat{\rho}_{\text{eq}}U(\gamma)^{-1}$, where $U(\gamma)$ is the unitary operator that implements the APD, and $F(\gamma) \equiv F[\hat{\rho}(\gamma)] = F[\hat{\rho}_{\text{eq}}] + O(\gamma^2)$, which is is minimized when $\gamma_{ab} = 0$. The free energy per flux quantum has the expansion

$$f(\gamma) = f(T,\nu) + \tfrac{1}{2}G^{abcd}(T,\nu)\gamma_{ab}\gamma_{cd} + O(\gamma^3), \tag{12}$$

where $G^{abcd} = G^{bacd} = G^{cdab}$. The "*guiding-center shear modulus*" (per flux quantum) of the state is given by $G^{ac}_{bd} = \epsilon_{be}\epsilon_{df}G^{aecf}$, with $G^{ac}_{bd} = G^{ca}_{db}$, and $G^{ac}_{bc} = 0$. (Note that in a spatially-covariant formalism, both stress $\sigma^a_b$ (the momentum current) and strain $\partial_c u^d$ (the gradient of the displacement field) are mixed-index tensors that are linearly related by the elastic modulus tensor $G^{ac}_{bd}$.) The entropy is left invariant by the APD, and the only affected term in the free energy is the correlation energy, which can be evaluated in terms of the deformed structure factor $s(\boldsymbol{q},\gamma)$, given by

$$s_\infty + \xi\int \frac{d^2\boldsymbol{q}'\ell_B^2}{2\pi} e^{i\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2}(s(\boldsymbol{q}') - s_\infty)e^{i\gamma^{ab}q_a q'_b\ell_B^2}, \tag{13}$$

with $\gamma^{ab} \equiv \epsilon^{ac}\epsilon^{bd}\gamma_{cd}$. This gives $G^{abcd}(T,\nu)\gamma_{ab}\gamma_{cd}$ as

$$\gamma_{ab}\gamma_{cd}\epsilon^{ae}\epsilon^{cf}\int \frac{d^2q\ell_B^2}{4\pi}\tilde{v}(\boldsymbol{q})q_e q_f s^{bd}(\boldsymbol{q};T,\nu). \tag{14}$$

Assuming only that the ground state $|\Psi_0\rangle$ of (2) has translational invariance, plus inversion symmetry (so it has vanishing electric dipole moment parallel to the 2D surface), GMP[1] used the SMA variational state $|\Psi(\boldsymbol{q})\rangle \propto \rho(\boldsymbol{q})|\Psi_0\rangle$ to obtain an upper bound $E(\boldsymbol{q}) \leq f(\boldsymbol{q})/s(\boldsymbol{q})$ to the energy of an excitation with momentum $\hbar\boldsymbol{q}$ (or electric dipole moment $e\boldsymbol{e}_a\epsilon^{ab}q_b\ell_B^2$), where

$$\begin{aligned} f(\boldsymbol{q}) &= \int \frac{d^2\boldsymbol{q}'\ell_B^2}{4\pi}\tilde{v}(\boldsymbol{q}')\left(2\sin\tfrac{1}{2}\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2\right)^2 s(\boldsymbol{q}',\boldsymbol{q}), \\ s(\boldsymbol{q}',\boldsymbol{q}) &\equiv \tfrac{1}{2}\left(s(\boldsymbol{q}'+\boldsymbol{q}) + s(\boldsymbol{q}'-\boldsymbol{q}) - 2s(\boldsymbol{q}')\right). \end{aligned} \tag{15}$$

Other than noting it was quartic in the small-$q$ limit, GMP did not not offer any further interpretation of $f(\boldsymbol{q})$. It can now be seen to have the long-wavelength behavior

$$\lim_{\lambda\to 0} f(\lambda\boldsymbol{q}) \to \tfrac{1}{2}\lambda^4 G^{abcd}q_a q_b q_c q_d\ell_B^4, \tag{16}$$

and is controlled by the guiding-center shear-modulus. Then the SMA result is, at long wavelengths, at $T = 0$,

$$E(\boldsymbol{q})s(\boldsymbol{q}) \leq \tfrac{1}{2}G^{abcd}q_a q_b q_c q_d\ell_B^2. \tag{17}$$

This relation also occurs (as an equality) in the phonon spectrum of the Wigner crystal of guiding centers[6]: if $\tilde{v}(\boldsymbol{q})$ diverges as $\boldsymbol{q} \to 0$, the leading behavior is

$$E(\boldsymbol{q}) \;\to\; \left(\tfrac{1}{2}G^{abcd}q_aq_bq_cq_d\ell_B^4\right)^{1/2} \times \left(\tilde{v}(\boldsymbol{q})\right)^{1/2}, \quad (18)$$

$$s(\boldsymbol{q}) \;\to\; \left(\tfrac{1}{2}G^{abcd}q_aq_bq_cq_d\ell_B^4\right)^{1/2} \times \left(\tilde{v}(\boldsymbol{q})\right)^{-1/2}. \quad (19)$$

In the case of the incompressible FQH fluid with a gap for excitations carrying an electric dipole moment, it appears[7] that the SMA may give the exact collective mode dispersion in the long-wavelength limit, in which case the FQH structure function, at least for one-component FQH fluids such as the Laughlin states, has the limit (at $T = 0$)

$$\lim_{\lambda\to 0} s(\lambda\boldsymbol{q}) \to \tfrac{1}{2}\lambda^4(G^{abcd}q_aq_bq_cq_d\ell_B^4)/E(0). \quad (20)$$

The $O(\lambda^4)$ behavior of $s(\lambda\boldsymbol{q})$ at small $\lambda$ is the fundamental property of FQH fluids identified by GMP.

The derivation above finally completes the picture of FQH incompressibility initiated by GMP, by linking the existence of the gap to the shear modulus of the fluid, and essentially to the geometry of flux attachment to its "elementary droplet" or "composite boson", and the energy cost of area-preserving diffeomorphisms that distort the shape of correlations around guiding centers. This key geometric ingredient appears to be missing in previously-proposed descriptions of FQH incompressibility ("Ginzburg-Landau-Chern-Simons theory" of a composite-boson superfluid[8], filling of "effective Landau levels" by composite fermions[9], or "non-commutative Chern-Simons theory"[5]), none of which derive directly from the Hamiltonian (2) and its algebra (4).

In the systems studied to date, the collective mode energy $E(\boldsymbol{q})$ has a "roton minimum" at finite $\boldsymbol{q} = \pm\boldsymbol{q}_{\min}$, with $2E(\boldsymbol{q}_{\min}) < E(0)$, so the lowest energy (quadrupolar) excitations at $\boldsymbol{q} = 0$ are either roton pairs, or a two-roton bound state[1], and the long-wavelength limit of the collective mode is hidden in the continuum, and presumably damped by decay into roton pairs. However, there seems no obvious reason why some choice of the interaction $\tilde{v}(\boldsymbol{q})$ could not expose $E(0)$ by moving it down below the bottom of the two-roton continuum.

I now outline the derivation of the duality (10). It will be convenient to impose periodic boundary conditions on a fundamental region of area $2\pi N_\Phi\ell_B^2$, so the allowed values of $\boldsymbol{q}$ define a Bravais lattice in reciprocal space with unit cell area $2\pi/N_\Phi\ell_B^2$. (Indeed, I initially noticed the duality in numerical exact-diagonalization results for $s(\boldsymbol{q})$ in such a geometry.) If both $\boldsymbol{q}$ and $\boldsymbol{q}'$ are allowed reciprocal vectors,

$$\left(e^{i\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2}\right)^{N_\Phi} = 1. \quad (21)$$

Then, for an allowed $\boldsymbol{q}$,

$$\left(e^{i\boldsymbol{q}\cdot\boldsymbol{R}}\right)^{N_\Phi}|\alpha\rangle = \left(\eta_{\boldsymbol{q}}\right)^{N_\Phi}|\alpha\rangle, \quad (22)$$

where $|\alpha\rangle$ is any of the $N_\Phi$ linearly-independent one-particle states in the Landau level, and $\eta(\boldsymbol{q})$ is 1 if $\frac{1}{2}\boldsymbol{q}$ is on the Bravais lattice, and $-1$ otherwise.

There is a recurrence relation, so that for $\boldsymbol{q}$ and $\boldsymbol{q}'$ on the Bravais lattice,

$$\rho(\boldsymbol{q} + N_\Phi\boldsymbol{q}') = \rho(\boldsymbol{q})\left(\eta_{\boldsymbol{q}'}e^{i\frac{1}{2}\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2}\right)^{N_\Phi} = \pm\rho(\boldsymbol{q}), \quad (23)$$

which means that there are exactly $(N_\Phi)^2$ linearly-independent generators $\rho(\boldsymbol{q})$, one of which is equivalent to $\rho(\boldsymbol{0})$, and which define a "Brillouin zone" (BZ). (The algebra (4) is now that of the generators of $U(N_\Phi)$.) The two-guiding-center exchange operator can be written as

$$P_{ij} = \frac{1}{N_\Phi}\sum_{\boldsymbol{q}'\in\mathrm{BZ}}{}' e^{i(\boldsymbol{q}'+\boldsymbol{q})\cdot\boldsymbol{R}_i}e^{-i(\boldsymbol{q}'+\boldsymbol{q})\cdot\boldsymbol{R}_j}, \quad (24)$$

where the primed sum is over *any* set of $(N_\Phi)^2$ reciprocal vectors $\{\boldsymbol{q}'\}$ that define a BZ. The freedom of choice of the BZ means that $\boldsymbol{q}$ in (24) can be chosen arbitrarily on the Bravais lattice. This expression can be rewritten as

$$P_{ij}e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_j} = \frac{1}{N_\Phi}\sum_{\boldsymbol{q}'\in\mathrm{BZ}}{}' e^{i\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2}e^{i\boldsymbol{q}'\cdot\boldsymbol{R}_i}e^{-i\boldsymbol{q}'\cdot\boldsymbol{R}_j}, \quad (25)$$

Take the expectation value, and sum over $i, j$, noting that $P_{ij} = \xi$ for $i \neq j$, $P_{ii} = N_\Phi$, and, for $\rho(\boldsymbol{q}) = \sum_i \exp i\boldsymbol{q}\cdot\boldsymbol{R}_i$ not equivalent to $\rho(\boldsymbol{0})$, that $\langle\rho(\boldsymbol{q})\rho(-\boldsymbol{q})\rangle = N_\Phi s(\boldsymbol{q})$, and $\langle N^2\rangle - \langle N\rangle^2 = N_\Phi s(0)$, $\nu = \langle N\rangle/N_\Phi$. Then

$$s(\boldsymbol{q}) - s_\infty = \frac{\xi}{N_\Phi}\sum_{q'\in\mathrm{BZ}}{}' e^{i\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2}\left(s(\boldsymbol{q}') - s_\infty\right). \quad (26)$$

No assumptions other than translational invariance (and full symmetry or antisymmetry under guiding-center exchange) of the density-matrix appear to have been made in this derivation. In particular, no rotational invariance has been assumed. Taking the thermodynamic limit $N_\Phi \to \infty$ leads directly to (10).

Further insight is provided by the expansion

$$\delta\rho(\lambda\boldsymbol{q}) = \delta N + i\lambda\epsilon^{ab}q_aK_b\ell_B^2 - \lambda^2q_aq_b\Lambda^{ab}\ell_B^2 + O(\lambda^3), \quad (27)$$

where $\delta N = N - \langle N\rangle$ commutes with $\delta\rho(\boldsymbol{q})$, and

$$[K_a, K_b] = 0, \quad [\Lambda^{ab}, K_c] = -\tfrac{i}{2}\left(\delta_c^a\epsilon^{bd} + \delta_c^b\epsilon^{ad}\right)K_d,$$
$$[\Lambda^{ab}, \Lambda^{cd}] = -\tfrac{i}{2}\left(\epsilon^{ac}\Lambda^{bd} + \epsilon^{ad}\Lambda^{bc} + a \leftrightarrow b\right). \quad (28)$$

Here $K_a$ are the generators of translations, $[K_a, \delta\rho(\boldsymbol{q})] = q_a\delta\rho(\boldsymbol{q})$, and $e\boldsymbol{e}_a\epsilon^{ab}K_b\ell_B^2$ is the total electric dipole moment. They annihilate an incompressible FQH state: $K_a|\Psi_0\rangle = 0$. The three independent components of the symmetric tensor $\Lambda^{ab}$ are the generators of linear APD's of the guiding centers, and obey the Lie algebra of the generators of the group $SL(2,R)$, isomorphic to $SO(2,1)$, with Casimir $\det\Lambda$[10]. Then at long wavelengths,

$$s(\lambda\boldsymbol{q}) \to \lambda^4\Gamma_S^{abcd}q_aq_bq_cq_d\ell_B^4, \quad (29)$$

where

$$\Gamma_S^{abcd} = (N_\Phi)^{-1}\left(\tfrac{1}{2}\langle\{\Lambda^{ab},\Lambda^{cd}\}\rangle - \langle\Lambda^{ab}\rangle\langle\Lambda^{cd}\rangle\right). \quad (30)$$

A bound to the tensor $\Gamma_S^{abcd}$ can be set using the related tensor

$$\begin{aligned}\Gamma_A^{abcd} &= (N_\Phi)^{-1}\tfrac{i}{2}\langle[\Lambda^{ab},\Lambda^{cd}]\rangle \\ &= \tfrac{1}{2}\left(\epsilon^{ac}\Gamma_H^{bd} + \epsilon^{ad}\Gamma_H^{bc} + a \leftrightarrow b\right), \quad (31)\\ \Gamma_H^{ab} &= \tfrac{1}{2}(N_\Phi)^{-1}\langle\Lambda^{ab}\rangle. \quad (32)\end{aligned}$$

Viewed as a $3\times 3$ Hermitian matrix, $M^{(ab),(cd)} = \Gamma_S^{abcd} \pm i\Gamma_A^{abcd}$ is positive (has no negative eigenvalues). This means $\Gamma_H^{ab}$ can set a lower bound to $\Gamma_S^{abcd}q_aq_bq_cq_d$[10].

Note that $\eta_H^{abcd} = (eB/2\pi)\Gamma_A^{abcd}$ is the guiding-center contribution[10] to the so-called dissipationless "Hall viscosity" tensor (there is also a Landau-orbit contribution[11]). Physically, $\Gamma_A^{abcd}$ gives the local stress (momentum current) induced by the linear response of the uniform FQH state to a non-uniform electric field:

$$\sigma_e^a(\boldsymbol{r}) = \frac{e}{2\pi}\epsilon_{eb}\left(\Gamma_A^{abcd} + \Gamma_A'^{abcd}\right)\partial_c E_d(\boldsymbol{r}), \quad (33)$$

where $\Gamma_A'^{abcd}$ in (33) is the Landau-orbit contribution[11].

If rotational invariance with a unimodular metric $g_{ab}$ is present, then guiding-center rotations are generated by $L = g_{ab}\Lambda^{ab}$, with $[H,L]=0$, and $\langle L^2\rangle - \langle L\rangle^2 = 0$, from which $\Gamma_S^{abcd} = \alpha\left(g^{ac}g^{bd} + g^{bc}g^{ad} - g^{ab}g^{cd}\right)$, and $s(\boldsymbol{q}) \to \alpha(g^{ab}q_aq_b\ell_B^2)^2$ at long wavelengths. Also $\Gamma_H^{ab} = \frac{1}{4}\bar{\ell}g^{ab}$, where $\bar{\ell} = \langle L\rangle/N_\Phi$, giving the bound $\alpha \geq \frac{1}{4}|\bar{\ell}|$.

Incompressible FQHE states may be partly classified by their "elementary droplet" or "composite boson" that consists of $p$ particles in $q$ orbitals ("flux attachment" $q$), where for $\nu = p/q$, $p$ and $q$ are the smallest integers obeying the statistical selection rule $(-1)^{pq} = \xi^p$ which ensures that pairs of droplets behave as bosons when exchanged. The elementary droplet also has a topologically-quantized[4] (integer or half-integer) "guiding-center spin" $\bar{s}$ (a 2D orbital spin unrelated to the Pauli spin of the electron), and $\bar{\ell} = \bar{s}/q$, with $|\bar{s}| < \frac{1}{2}pq$. If there is rotational invariance, and $\bar{N}$ elementary droplets are combined to form a circular droplet of FQH fluid centered at the origin, with no internal excitations, the total guiding-center angular momentum is

$$L_0 = \tfrac{1}{2}pq\bar{N}^2 + \bar{s}\bar{N}, \quad \frac{g_{ab}}{2\ell_B^2}\sum_i R_i^aR_i^b|\Psi_0\rangle = L_0|\Psi_0\rangle. \quad (34)$$

The first term in $L_0$ is the uniform-disk contribution $\frac{1}{2}\nu(N_\Phi)^2$, $N_\Phi = q\bar{N}$; the second term is $\bar{\ell}N_\phi$[12].

The bound $\alpha \geq |\bar{s}|/4q$ is satisfied as an equality[1, 10] in the case of certain polynomial model wavefunctions (Laughlin, Moore-Read, *etc.*) that can be viewed as correlators of chiral conformal field theories, and have rotational invariance. Read and Rezayi[13] have argued not only that the bound satisfied by these model states will remain satisfied as an equality in the presence of perturbations that maintain rotational invariance, but also claim that this must be be generically true for "all lowest-Landau level" FQH states with rotational invariance. The counter-propagating principal hierarchy/Jain sequence $\nu = n/(4n-1)$, $n = 1, 2, \ldots$, descending from the $\nu = \frac{1}{3}$ Laughlin state, with $p = n$, $q = 4n-1$, has $\bar{s} = \frac{1}{2}n(n-3)$. If the claim of Ref.[13] were generally correct, the $\nu = \frac{3}{11}$ state would have vanishing $\alpha$. Instead, satisfaction of the bound as an equality is likely to occur only in *maximally-chiral* FQH states without counter-propagating components (in this sequence, $n = 1$ only). The property of maximal chirality of a FQH state can in principle be identified from its entanglement spectrum[14], without reference to model wavefunctions.

In summary, I have exhibited a self-duality of the guiding-center structure function $s(\boldsymbol{q})$ of translationally-invariant states of a partially-filled Landau level (a "quantum geometry" problem) and used it to interpret the long-wavelength collective excitations of FQH states in terms of a shear modulus. Bounds on the $O(q^4)$ behavior of $s(\boldsymbol{q})$ were also obtained.

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